FCDs for robust regression model

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Here are the full conditional distributions (FCD's) for this problem:

- For β_0:

$$p(\beta_0|\ldots) \propto \left[\prod_i N\left(y_i|\beta_0+\beta_1 x_i, \frac{\sigma^2}{\tau_i}\right)\right] N(\beta_0|0,10)$$

$$\propto \left[\prod_i \exp\left(-\frac{\tau_i}{2\sigma^2}(y_i-\beta_1 x_i-\beta_0)^2\right)\right] \exp\left(-\frac{1}{2\times 10}\beta_0^2\right)$$

$$\propto \exp\left(-\frac{1}{2\sigma^2}\sum_i \tau_i(-2\beta_0[y_i-\beta_1 x_i]+\beta_0^2)\right) \exp\left(-\frac{1}{2\times 10}\beta_0^2\right)$$

$$\propto \exp\left(-\frac{1}{2}\left\{\beta_0^2\left[\frac{1}{\sigma^2}\sum_i \tau_i+\frac{1}{10}\right]-2\beta_0\left(\frac{1}{\sigma^2}\right)\sum_i \tau_i[y_i-\beta_1 x_i]\right\}\right)$$

$$p(\beta_0|\ldots) = N\left(\left[\frac{1}{\sigma^2}\sum_i \tau_i+\frac{1}{10}\right]^{-1}\frac{1}{\sigma^2}\sum_i \tau_i[y_i-\beta_1 x_i], \left[\frac{1}{\sigma^2}\sum_i \tau_i+\frac{1}{10}\right]^{-1}\right)$$

- For β_1:

$$p(\beta_1|\ldots) \propto \left[\prod_i N\left(y_i|\beta_0+\beta_1 x_i, \frac{\sigma^2}{\tau_i}\right)\right] N(\beta_1|0,10)$$

$$\propto \left[\prod_i \exp\left(-\frac{\tau_i}{2\sigma^2}(y_i-\beta_0-\beta_1 x_i)^2\right)\right] \exp\left(-\frac{1}{2\times 10}\beta_1^2\right)$$

$$\propto \exp\left(-\frac{1}{2}\frac{1}{\sigma^2}\sum_i \tau_i\left(-2\beta_1 x_i[y_i-\beta_0]+\beta_1^2 x_i^2\right)\right) \exp\left(-\frac{1}{2\times 10}\beta_1^2\right)$$

$$\propto \exp\left(-\frac{1}{2}\left[\beta_1^2\left[\frac{1}{\sigma^2}\sum_i \tau_i x_i^2+\frac{1}{10}\right]-2\beta_1\frac{1}{\sigma^2}\sum_i \tau_i x_i[y_i-\beta_0]\right]\right)$$

$$p(\beta_1|\ldots) = N\left(\left[\frac{1}{\sigma^2}\sum_i \tau_i x_i^2+\frac{1}{10}\right]^{-1}\frac{1}{\sigma^2}\sum_i \tau_i x_i[y_i-\beta_0], \left[\frac{1}{\sigma^2}\sum_i \tau_i x_i^2+\frac{1}{10}\right]^{-1}\right)$$

- For τ_i:

$$p(\tau_i|\ldots) \propto N\left(y_i|\beta_0+\beta_1 x_i, \frac{\sigma^2}{\tau_i}\right) Gamma\left(\tau_i|\frac{v}{2},\frac{v}{2}\right)$$

$$\propto \frac{1}{\sqrt{\frac{1}{\tau_i}}} \exp\left(-\frac{\tau_i}{2\sigma^2}(y_i - \beta_0 - \beta_1 x_i)^2\right) \tau_i^{\frac{v}{2}-1} \exp\left(-\tau_i\left(\frac{v}{2}\right)\right)$$

$$\propto \tau_i^{\frac{1+v}{2}-1} \exp\left(-\tau_i\left[\frac{1}{2\sigma^2}(y_i - \beta_0 - \beta_1 x_i)^2 + \frac{v}{2}\right]\right)$$

$$p(\tau_i| \ldots) = Gamma\left(\frac{1+v}{2}, \frac{\frac{1}{\sigma^2}(y_i - \beta_0 - \beta_1 x_i)^2 + v}{2}\right)$$

- For v:

$$p(v| \ldots) \propto \left[\prod_i Gamma\left(\tau_i | \frac{v}{2}, \frac{v}{2}\right)\right] Unif(v|0{,}50)$$

To sample v, we rely on a Metropolis-Hastings algorithm for which we propose v from a normal distribution. If we end up proposing negative values, we just reflect it back to being positive by taking its absolute value. If we get values that are above 50, we reflect these values back into the 0-50 interval.

- For σ^2:

$$p\left(\frac{1}{\sigma^2}\middle| \ldots\right) \propto \left[\prod_i N\left(y_i | \beta_0 + \beta_1 x_i, \frac{\sigma^2}{\tau_i}\right)\right] Gamma\left(\frac{1}{\sigma^2}\middle| a, b\right)$$

$$\propto \left[\prod_i \left(\frac{1}{\sigma^2}\right)^{\frac{1}{2}} \exp\left(-\frac{\tau_i}{2\sigma^2}(y_i - \beta_0 - \beta_1 x_i)^2\right)\right] \left(\frac{1}{\sigma^2}\right)^{a-1} \exp\left(-b\frac{1}{\sigma^2}\right)$$

$$\propto \left(\frac{1}{\sigma^2}\right)^{\frac{N+2a}{2}-1} \exp\left(-\frac{1}{\sigma^2}\left[\frac{1}{2}\sum_i \tau_i (y_i - \beta_0 - \beta_1 x_i)^2 + b\right]\right)$$

$$p\left(\frac{1}{\sigma^2}\middle| \ldots\right) = Gamma\left(\frac{N+2a}{2}, \frac{1}{2}\sum_i \tau_i (y_i - \beta_0 - \beta_1 x_i)^2 + b\right)$$